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MANAGING ERROR ESTIMATES FOR SEMIDISCRETE FINITE ELEMENT APPROXIMATIONS OF SEMILINEAR PARABOLIC EQUATIONS IN A NONCONVEX POLYGON

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ABSTRACT
In this paper, we consider the semilinear parabolic problems with homogeneous Dirichlet boundary conditions in a two-dimensional nonconvex polygon. We study the semidiscrete error analysis which is based on an error splitting technique together with rigorous regularity analysis of the semilinear parabolic equations. Previously, Chatzipantelidis et al. [BIT Numer. Math., 46 (2006), pp. S113-S143] made an effort to analyze for problems in nonconvex polygons mainly focused on linear models. A special feature in a nonconvex polygon is the presence of singularities in the solutions generated by the corners. Due to the nonlinearity in the forcing term and the non-smoothness of the solution in a nonconvex polygon, the analysis is not straightforward. We establish the convergence in $L^\infty(L^2)$ *and* $L^\infty(H^1)$ *for the semidiscrete finite element solution.*

Keywords:- *Semilinear parabolic problem, nonconvex polygon, singularity, error estimates*

AMS subject classifications— 65M60, 65N15, 65N30

INTRODUCTION

Numerical methods and analysis for linear models defined in a nonconvex polygonal domain have been investigated extensively in the last several years (cf. [5], [6]). The purpose of this paper is to study certain error estimates for piecewise linear finite element approximations to solutions of the semilinear parabolic equations in a nonconvex polygonal domain.

Let Ω be a bounded nonconvex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We restrict our attention to spatially semidiscrete approximate solutions of the semilinear initial-boundary value problem, for $u = u(x,t)$,

$$\begin{aligned} &u_t - \Delta u = f(u) \quad \text{in } \Omega,\ t \in J, \\ &u = 0 \quad \text{on } \partial\Omega,\ t \in J, \qquad (1) \\ &u(\cdot\ ,0) = v \quad \text{in } \Omega, \end{aligned}$$

where u_t denotes $\partial u/\partial t$, the Laplacian denoted by $\Delta = \sum_{j=1}^{2} \partial^2/\partial x_j^2$ and $J = (0,T]$, $T > 0$, be a finite interval in time. We assume the smooth function f on $\mathbb{R}$ such that

$$|f'(u)| \le B \quad \text{for } u \in \mathbb{R}. \qquad (2)$$

The solution of parabolic partial differential equations in nonconvex polygonal domains is involved in many physical applications such as heat conduction in chip design, environment modeling, porous media flow and modeling of complex technical engines (cf. [8]). The analysis for such PDEs and for the corresponding numerical methods is always been a challenging research area due to the non-smoothness of the solution around the reentrant corner of the domain. For simplicity, we assume that exactly one interior angle ω is reentrant, i.e., such that $\pi < \omega < 2\pi$. Setting $\beta = \pi / \omega$, we have $1/2 < \beta < 1$. For the special case of an L-shaped domain, $\omega = 3\pi/2$ and $\beta = 2/3$.

The regularity of the solutions of a simple elliptic problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \partial\Omega, \qquad (3)$$

for the nonconvex domain has been extensively studied, see Grisvard [[9], [10]]. In [11], Kellogg have shown the regularity shift-theorem for the solution of the problem (3) as

$$\| u \|_{H^{1+s}} \le C \| f \|_{H^{-1+s}} = C \| \Delta u \|_{H^{-1+s}} \quad \text{for } 0 \le s \le \beta,$$

(4)

where the $H^s = H^s(\Omega)$ are fractional order Sobolev spaces, see Section II. But we can not expect such estimate (4) for $s \ge \beta$ due to the singularity in the solution. A more precise analysis for the case $s = \beta$ was presented by Bacuta et al. [4] in the framework of Besov spaces. We have studied the semidiscrete finite element methods (FEM) and derive convergence properties in both the $L^\infty(L^2)$ and $L^\infty(H^1)$ norms. To the best of our knowledge there is no literature available concerning finite element method for semilinear parabolic problems in nonconvex domain. The rest of the paper is organized as follows. In Section II we have presented the existence and uniqueness of the semidiscrete finite element solution for the problem (1). In this section we have stated our main result for the estimates of the error between the solutions of the continuous and the semidiscrete problem. Section III devoted to the proof of the semidiscrete error estimates. There is a reduction in the convergence rate from optimal order to $O(h^{2\beta})$ caused by the presence of singularity in the solution due to the reentrant corner in the domain. However a systematical mesh refinement near the corners have been introduced in this section which gives an improvement of the convergence rate to the optimal order. Finally, some concluding remarks are presented in the last section.

NOTATIONS AND PRELIMINARIES

In this section, we introduce some basic preliminary notations which will be used throughout the paper. We denote the standard Lebesgue spaces by $L^p(\Omega)$, $1 \le p \le \infty$, with the norm $\| . \|_{L^p(\Omega)}$. In particular, for $p = 2$, $L^2(\Omega)$ is a Hilbert space with the norm $\| . \| = \| . \|_{L^2(\Omega)}$ induced by the inner product $(u, v) = \int_\Omega u(x) v(x) dx$. For an integer $m > 0$ and $1 \le p < \infty$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space. In particular, for $p = 2$, we denote the Hilbert space $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with the norm $\| . \|_{H^m(\Omega)}$ (cf. [1], [2]). For an integer $m \ge 0$, set $s = m + \sigma$, $0 < \sigma < 1$, and then $H^s = H^s(\Omega)$ denote the Sobolev spaces of fractional order with the norm defined by

$$\| u \|_{H^s} = (\| u \|^2_{H^m} + \sum_{|\alpha| = m} \iint_{\Omega \times \Omega} \frac{| D^\alpha u(x) - D^\alpha u(y) |^2}{| x - y |^{2+2\sigma}} dx dy)^{1/2}.$$

For a given Banach space B and for $1 \le p < \infty$, we define

$$L^p(0,T;B) = \{ v : [0,T] \to B \mid v(t) \in B \quad \text{for almost all } t \in [0,T] \text{ and } \int_0^T \| v(t) \|_B^p \, dt < \infty \}$$

equipped with the norm

$$\| v \|_{L^p(0,T;B)} := (\int_0^T \| v(t) \|_B^p)^{1/p},$$

with the standard modification for $p = \infty$. We write

$$\| v \|_{L^p(0,T;B)} = \| v \|_{L^p(B)}.$$

Semidiscrete finite element approximation

Let $T_h = \{K\}$ be the family of quasiuniform triangulations of Ω with $\max_{K\in T_h} diam(k) \leq h$, in the sense of Ciarlet [7] and Thomée [12]. Let thus $S_h \subset H_0^1(\Omega)$ be the finite dimensional space corresponding to the triangulations T_h is defined by

$S_h = \{\chi \in C : \chi|_K$ is linear, $\forall K \in T_h$ and $\chi|_{\partial\Omega} = 0\}$,

where $C = C(\Omega)$ be the space of continuous functions $\overline{\Omega}$. We study the semidiscrete solution $u_h : \bar{J} \to S_h$ such that

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla\chi) = (f(u_h), \chi)$$

$$\forall \chi \in S_h,\ t \in J, \qquad (5)$$

with $u_h(0) = v_h$,

where $v_h \in S_h$ is an approximation of v. We represent the solution as $u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\varphi_j(x)$, where $\{\varphi\}_{j=1}^{N_h}$ is the standard nodal basis functions for $S_{h,}$ thus (5) may be written as

$$\sum_{j=1}^{N_h} \alpha_j'(t)(\varphi_j, \varphi_k) + \sum_{j=1}^{N_h} \alpha_j(t)(\nabla\varphi_j, \nabla\varphi_k)$$

$$= (f(\sum_{l=1}^{N_h} \alpha_l(t)\varphi_l), \varphi_k),$$

with $\alpha_j(0) = \gamma_j$ for $j,k = 1,2,\cdots,N_h$,

where γ_j the components of the given initial approximation of v_h and $N_h = \dim(S_h)$. In matrix notation this may be written as

$A\alpha'(t) + B\alpha(t) = \tilde{f}(\alpha)$, for $t \in J$,

with $\alpha(0) = \gamma$,

Where $A = (a_{jk})$, $B = (b_{jk})$ with elements $a_{jk} = (\varphi_j, \varphi_k)$ and $b_{jk} = (\nabla\varphi_j, \nabla\varphi_k)$, $\tilde{f}(\alpha) = (f_k(\alpha))$ be the vector with entries $(f_k(\alpha)) = (f(\sum_{l=1}^{N_h} \alpha_l\varphi_l), \varphi_k)$ and $\gamma = (\gamma_k)$. Since the mass matrix A and the stiffness matrix B are positive definite and hence in particular, A is invertible. Also $\tilde{f}(\alpha)$ are globally Lipschitz continuous on $\mathbb{R}^{N_h}$. Therefore the system has a unique solution for $t \in J$.

The main aim of this paper is to prove the following estimate in $L^\infty(L^2)$ and $L^\infty(H^1)$ for the error between the solutions of the semidiscrete problem (5) and the continuous problem (1).

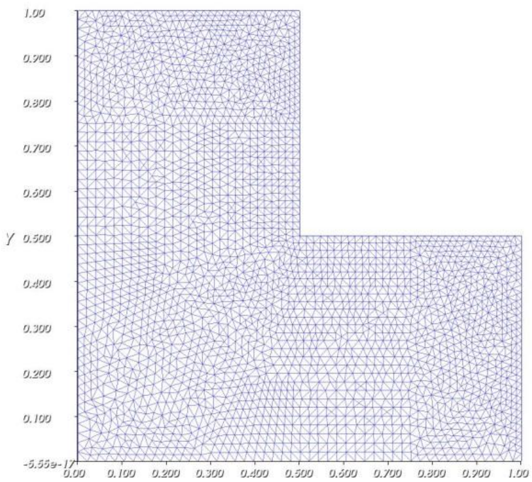

Figure 1: Finite element discretizations for the L-shaped domain, #triangles= 5178 and #dof= 2702.

Theorem 1. *Let* u_h *and* u *be the solutions of (5) and (1), respectively. Then, under the appropriate regularity assumptions for* u, *and for* $\beta < s < 1$, $t \in \overline{J}$, *we have*

$$\| u_h(t) - u(t) \| \le C \| v_h - v \| + C(u) h^{2\beta}, \qquad (6)$$

and

$$\| \nabla (u_h(t) - u(t)) \| \le C \| \nabla (v_h - v) \| + C(u) h^{\beta}. \qquad (7)$$

For the purpose of the proof of Theorem 1, we introduce the so called *elliptic* or *Ritz projection* R_h onto $S_{h,}$, defined by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi) \quad \forall \chi \in S_{h,} \quad \text{for} \quad v \in H_0^1(\Omega). \ (8)$$

Setting $\chi = R_h v$ in (8), it follows that the Ritz projection is stable in $H_0^1(\Omega)$, i.e.,

$$\| \nabla R_h v \| \ \le \ \| \nabla v \| \qquad \forall v \in H_0^1(\Omega).$$

We therefore have the following error estimate in this projection.

Lemma 2. *Let* R_h *be defined by (8). Then for* $v \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$, $\beta < s \le 1$ *with* $C = C_{s,}$ *we have*

$$\| R_h v - v \| + h^{\beta} \| \nabla (R_h v - v) \| \le C h^{2\beta} \| \Delta v \|_{H^{-1+s}}.$$

Proof. The proof is easily follows from [5, Lemma 2.5].

PROOF OF THEOREM 1

We first decompose the error in a standard way as

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho, \qquad (9)$$

where R_h is defined by (9). Since ρ is bounded in view of Lemma 2, hence it only remains to estimate θ. We have using (9), for $\chi \in S_{h,}$,

$$\begin{aligned}
&(\theta_t, \chi) + (\nabla \theta, \nabla \chi) \\
&= (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) - (R_h u_t, \chi) - (\nabla R_h u, \nabla \chi) \\
&= (f(u_h), \chi) - (R_h u_t, \chi) - (\nabla u, \nabla \chi) \\
&= (f(u_h), \chi) - (R_h u_t - u_t, \chi) - (u_t, \chi) - (\nabla u, \nabla \chi) \\
&= (f(u_h) - f(u), \chi) - (\rho_t, \chi),
\end{aligned}$$

Or,

$$(\theta_t, \chi) + (\nabla \theta, \nabla \chi) = (f(u_h) - f(u), \chi) - (\rho_t, \chi). \ (10)$$

Therefore, choosing $\chi = \theta$ and using (2) and (9), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \| \theta \|^2 + \| \nabla \theta \|^2 \\
&\le C(\| u_h - u \| . \| \theta \| + \| \rho_t \| . \| \theta \|) \\
&\le C((\| \theta \| + \| \rho \|). \| \theta \| + \| \rho_t \| . \| \theta \|) \\
&\le C(\| \theta \|^2 + \| \rho \|^2 + \| \rho_t \|^2) + \| \nabla \theta \|^2,
\end{aligned}$$

where we have used Friedrich's inequality $\| \theta \| \le c \| \nabla \theta \|$. After integration, this yields

$$\| \theta(t) \|^2 \le \| \theta(0) \|^2 + C \int_0^t (\| \theta \|^2 + \| \rho \|^2 + \| \rho_t \|^2) ds,$$

applying Gronwall's lemma this leads to

$$\| \theta(t) \|^2 \le C \| \theta(0) \|^2 + C \int_0^t (\| \rho \|^2 + \| \rho_t \|^2) ds, \ (11)$$

where C now depends on T. Now, using Lemma 2 we have

$$\| \theta(0) \| \le \| v_h - v \| + \| R_h v - v \|$$

$\leq \| v_h - v \| + Ch^{2\beta} \| \Delta v \|_{H^{-1+s}}$.
Hence, in view of Lemma 2, (11) yields
$\| \theta(t) \| \leq C \| v_h - v \| + C(u)h^{2\beta}$,
which completes the proof of (6).
In order to show the estimate for the gradient in (7), we split the error term as before
$\nabla(u_h(t) - u(t)) = \nabla\theta(t) + \nabla\rho(t)$.
Here $\nabla\rho(t)$ is bounded as claimed in Lemma 2, we only need to estimate $\nabla\theta(t)$. This time with $\chi = \theta_t$ in (10), we have

$$\| \theta_t \|^2 + \frac{1}{2}\frac{d}{dt} \| \nabla\theta \|^2$$
$$\leq C(\| u_h - u \| . \| \theta_t \| + \| \rho_t \| . \| \theta_t \|)$$
$$\leq C((\| \theta \| + \| \rho \|). \| \theta_t \| + \| \rho_t \| . \| \theta_t \|)$$
$$\leq C(\| \theta \|^2 + \| \rho \|^2 + \| \rho_t \|^2) + \frac{1}{2} \| \theta_t \|^2 .$$

After integration and using the Friedrich's inequality $\| \theta \| \leq c \| \nabla\theta \|$, this leads to

$$\| \nabla\theta(t) \|^2$$
$$\leq \| \nabla\theta(0) \|^2 + C\int_0^t (\| \nabla\theta \|^2 + \| \rho \|^2 + \| \rho_t \|^2) ds,$$

applying Gronwall's lemma (with C now depends on T), this gives

$$\| \nabla\theta(t) \|^2 \leq C \| \nabla\theta(0) \|^2 + C\int_0^t (\| \rho \|^2 + \| \rho_t \|^2) ds,$$

using Lemma 2 together with the estimate of $\| \nabla\theta(0) \|$, we obtain
$\| \nabla\theta(t) \| \leq C \| \nabla(v_h - v) \| + C(u)h^{\beta}$.
This completes the proof of the theorem. □

Remark 3. Note that, $O(h^{2\beta})$ is the best possible convergence we obtain away from the nonconvex corner as the singularity at the reentrant corner pollutes the finite element solution everywhere in Ω for the case of globally quasiuniform mesh. However, with a systematical refinement of triangulations towards the nonconvex corner we obtain an optimal order convergence $O(h^2)$ and $O(h)$ in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms, respectively. The refinement were introduced by Babuška [3].
Further refinements towards the nonconvex corner. In order to introduce the refinement of triangulations systematically (cf. [5]), let $d(x)$ be the distance to the nonconvex corner and $d_j = 2^{-j}$, for $j = 0,1,..., \hat{J}$. Assume that, for $j = 0,1,..., \hat{J}$,

$$\Omega_j = \{x \in \Omega : d_j / 2 \leq d(x) \leq d_j\},$$
$$\Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1},$$
$$\Omega_I = \{x \in \Omega : d(x) \leq d_{\hat{j}} / 2\}.$$

Choose $\hat{J}$ such that $d_{\hat{j}} \approx h^{1/\beta}$, where h be the meshsize in the interior of the domain. Furthermore, choose $\gamma \geq 1/\beta$ such that
$h_j \leq Chd_j^{1-\beta+\varepsilon}$ and $ch^{\gamma} \leq h_I \leq Ch^{1/\beta}$, with $c > 0$, (12)

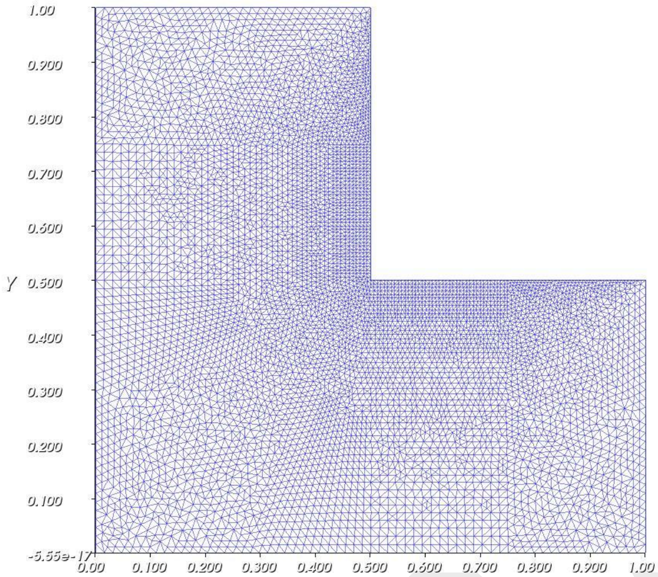

Figure 2: Further refinements made towards the nonconvex corner for the L-shaped domain, #triangles= 8646, #dof= 4468.

where ε be any small positive number, and h_j denotes the maximal meshsize on Ω_j. Also let the mesh is locally quasiuniform on each Ω'_j so that $h_{\min} \geq h^{\gamma}$ and $\dim(S_h) \leq Ch^{-2}$. The finite element triangulations for an $L-$shaped domain are depicted in Fig. 1 and Fig. 2.

We now have the following auxiliary result.

Lemma 4. Let R_h be defined by (8). Then with the triangulations above, satisfying (12), we have

$$\| R_h v - v \| + h \| \nabla (R_h v - v) \| \leq Ch^2 \| \Delta v \| .$$

Proof. Following Chatzipantelidis et al. [5, Lemma 2.9] with $s = 1$, the proof is easily follows.

We finally show that the optimal order error bounds for Theorem 1 is obtained by refinements towards the nonconvex corner.

Theorem 5. Let u_h and u be the solutions of (5) and (1), respectively. Assume that the triangulations underlying the S_h are refined as in Lemma 4. Then, if v_h is appropriately chosen, we have, with $C = C(u, T)$,

$$\| u_h(t) - u(t) \| + h \| \nabla (u_h(t) - u(t)) \| \leq Ch^2$$

for $t \in \bar{J}$.

Proof. In view of Lemma 4 and following the similar argument as in the proof of Theorem 1, the rest of the proof is standard.

CONCLUSIONS

We have presented an approach for the solution of Semilinear parabolic equations in nonconvex polygonal domains. A priori error estimates in $L^{\infty}(L^2)$ and $L^{\infty}(H^1)$ norms for the semidiscrete case are discussed and analyzed. Starting from a convergence rate $O(h^{2\beta})$ and $O(h^{\beta})$ in $L^{\infty}(L^2)$ and $L^{\infty}(H^1)$, respectively, for the nonconvex polygon we have obtained an optimal order convergence in both the norms with a proper mesh refinement near the reentrant corners of the domain.

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